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Surface-wave generation revisited

By JOHN MILES

Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, CA 92093-0225, USA

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The quasi-laminar model for the transfer of energy to a surface wave from a turbulent shear flow (Miles 1957) is modified to incorporate the wave-induced perturbations of the Reynolds stresses, which are related to the wave-induced velocity field through the Boussinesq closure hypothesis and the ancillary hypothesis that the eddy viscosity is conserved along streamlines. It is assumed that the basic mean velocity is $U(z) = (U_*/\kappa)\log(z/z_0)$ for sufficiently large z (elevation above the level interface) and that $U(z_1) \gg U_*$ for $kz_1 = O(1)$, where k is the wavenumber. The resulting vorticity-transport equation is reduced, through the neglect of diffusion, to a modification of Rayleigh's equation for wave motion in an inviscid shear flow. The energy transfer to the surface wave, which comprises independent contributions from the critical layer (where $U = c$, the wave speed) and the wave-induced Reynolds stresses, is calculated through a variational approximation and, independently, through matched asymptotic expansions. The critical-layer component is equivalent to that for the quasi-laminar model. The Reynolds-stress component is similar to, but differs quantitatively from, that obtained by Knight (1977), Jacobs (1987) and van Duin & Janssen (1992). The predicted energy transfer agrees with the observational data compiled by Plant (1982) for $1 \lesssim c/U_* \lesssim 20$, but the validity of the logarithmic profile for the calculation of the energy transfer in the critical layer for $c/U_* < 5$ remains uncertain. The basic model is unreliable (for water waves) if $c/U_* \lesssim 1$, but this domain is of limited oceanographic importance. It is suggested that Kelvin–Helmholtz instability of air blowing over oil should provide a good experimental test of the present Reynolds-stress modelling and that this modelling may be relevant in other geophysical contexts.

1. Introduction

Some thirty-five years ago, I constructed a quasi-laminar model (Miles 1957, hereinafter referred to as I) for the transfer of energy from a shear flow to a surface wave in which turbulence is implicitly included through a prescribed velocity profile but viscous stresses and the wave-induced Reynolds stresses are neglected.† Viscous stresses have been incorporated in this model (Benjamin 1959; Miles 1959*a*) and are important for those relatively short waves for which resonance with Tollmien–Schlichting waves is possible (Miles 1962), but they are negligible for the long waves to be considered here. The incorporation of the wave-induced Reynolds stresses in the equations of motion is straightforward (see the Appendix in I), but their expression in terms of the wave-induced velocity field requires *ad hoc* modelling for which direct observational confirmation is as yet unavailable. Many such models have been

† See Phillips (1957, 1977) and Miles (1960) for the development of a complementary model that describes the direct effects of turbulent fluctuations in pressure on the surface wave and is relevant for the initial stages of wave generation.

developed during the past twenty-five years; guided by these models, I revisit the problem with the purpose of incorporating the wave-induced Reynolds stresses in a modification of the Rayleigh equation that governs the quasi-laminar model.

I consider a turbulent shear flow with the mean velocity profile $U(z)$ over the wave

$$z = a\cos kx \equiv h_0(x) \quad (ka \ll 1) \tag{1.1}$$

in a reference frame moving in the x-direction with the wave speed $c = (g/k)^{\frac{1}{2}}$. The formulation may be extended to an obliquely moving wave through Squire's transformation (Drazin & Reid 1981). I assume that

$$U(z) = U_1 \ln(z/z_0), \quad U_1 \equiv U_*/\kappa \tag{1.2a, b}$$

(U_* is the friction velocity and κ is von Kármán's constant) for sufficiently large z,

$$ka \ll 1, \quad kz_c \ll 1, \quad kD \gg 1, \quad \epsilon \equiv \frac{U_1}{V} < \frac{U_1}{V-c} \ll 1, \tag{1.3a–d}$$

where z_c is the elevation of the critical layer, at which $U(z_c) = c$, D is the depth of the turbulent boundary layer, $V \equiv U(z_1)$, and $kz_1 = O(1)$. The profile (1.2) may be continued to the surface-wave interface, where $U = 0$, by replacing z by $z+z_0$, which implies $U \to U_1 z/z_0$ as $z \downarrow 0$; however, the present formulation is valid for any constant-stress ($\nu U' = U_*^2$ for $z > 0$) boundary layer.

In the quasi-laminar model of I, the energy transfer to the surface wave is associated with a singularity at the critical layer and is described by

$$(kc\bar{E})^{-1}(\partial\bar{E}/\partial t) = s\beta(U_1/c)^2 \equiv \sigma, \quad \beta = -\pi(U_c''/kU_c')(\overline{w_c^2}/U_1^2\overline{h_0^2}) \equiv \beta_c, \tag{1.4a, b}$$

where E is the wave energy, the overbar signifies an average over x, $s \equiv \rho_a/\rho_w$ is the air–water density ratio, U_1 is the reference velocity (1.2b), the subscript c implies $z = z_c$, and w_c is the vertical velocity at the critical layer.

If the wave-induced turbulent fluctuations are retained in the equations of motion and the $O(ka)$ components thereof averaged in the transverse (y) direction, the transport equation for the wave-induced vorticity ω (2.4) may be placed in the form (Miles 1967)

$$\mathrm{D}\omega/\mathrm{D}t = \partial_x\partial_z\langle w'^2 - u'^2\rangle + (\partial_x^2 - \partial_z^2)\langle u'w'\rangle \equiv R, \tag{1.5}$$

where u' and w' are the x- and z-components of the turbulent velocity fluctuation, and $\langle\ \rangle$ signifies an average over y. If R is neglected, as in the quasi-laminar model, (1.5) is equivalent to Rayleigh's equation, and its solution leads to (1.4). If $R \neq 0$ and $\epsilon \ll 1$ (1.4b) remains valid for the energy transfer associated with the critical layer, but there is an additional energy transfer associated with the wave-induced Reynolds stresses.

Further progress requires a closure hypothesis for the determination of R. Attempts (see Miles 1967) to express R directly in terms of the wave-induced vorticity through mixing-length or similarity arguments were unsuccessful, and it seems clear in retrospect that any first-order closure model should start from the Reynolds-stress tensor. The first such models were those of Davis (1972, 1974) and Townsend (1972). These were followed by the studies of Chalikov (1976, 1978), Gent & Taylor (1976), Gent (1977) and Al-Zanaidi & Hui (1984), all of which (like those of Davis and Townsend) culminate in numerical integrations, and those of Knight (1977), Jacobs (1987) and van Duin & Janssen (1992), who invoke the Boussinesq closure hypothesis†

† This description for the direct construction of the Reynolds-stress tensor appears to enjoy wide acceptance in the current literature of turbulence, but it is worth recalling that Boussinesq (1877) simply posited an eddy viscosity in the Navier–Stokes equations without any direct reference to turbulent fluctuations or Reynolds stresses.

and construct analytical solutions through matched asymptotic expansions to obtain (cf. (1.4))

$$\beta_\nu = 2\kappa^2\left(\frac{V-c}{U_1}\right), \quad V = U(1/k), \tag{1.6a, b}$$

for the energy-transfer coefficient (defined as in (1.4*a*)) for the logarithmic profile (1.2). Each of these models has provided valuable insights and bases for further progress, but only those of Jacobs and van Duin & Janssen come close to the desideratum of a counterpart for the Orr–Sommerfeld equation or, absent diffusion, Rayleigh's equation.

The simplicity of Knight's result (1.6) suggests a more direct connection with the underlying physics than is provided by his matched-asymptotic solution or those of Jacobs and van Duin & Janssen. Moreover, I find, through a variational approximation (§5) and, independently, through matched asymptotic expansions (Appendix C), that their reference velocity (1.6*b*) should be replaced by

$$V \equiv U(z_1) = 2k\int_0^\infty e^{-2kz}U(z)\,dz, \quad kz_1 = \tfrac{1}{2}e^{-\gamma} = 0.281 \tag{1.7a, b}$$

($\gamma = 0.577$ is Euler's constant). The adoption of $V = U(1/k)$, rather than $U(z_1)$, yields a value of β_ν that is too high by $-2\kappa^2(\gamma+\ln 2) = 0.406$.

Each of Knight, Jacobs and van Duin & Janssen neglects the energy transfer associated with the phase shift across $U = c$, which is given by (1.4) in the present approximation. This neglect is justified if the critical layer ($z = z_c$) lies within a sublayer of negligible profile curvature, but this condition is not satisfied throughout the parametric domain ($\epsilon \ll 1$) in which their models are viable.

The Boussinesq closure hypothesis must be supplemented by the explicit prescription of an eddy viscosity ν. This phenomenological construct is given by $\nu = U_*^2/U'$ in a constant-stress boundary layer and by $\kappa U_* z$ for the logarithmic profile (1.2), but some additional hypothesis is required for its determination in the perturbed flow. Knight (1977) adopts Saffman's (1970) two-equation model of turbulent flow, which contains five empirical (but constrained) parameters; however, Knight's approximation (1.6) is independent of these parameters. Jacobs (1987) generalizes $\nu = \kappa U_* z$ for the basic flow (1.2) by positing (cf. Davis 1970) $\nu = U_* \ell_m$, where $\ell_m = \kappa(z-h_0)$ is a mixing length and $z-h_0$ is the elevation above the surface wave. Van Duin & Janssen (1992) include a viscous sublayer (as also do Al-Zanaidi & Hui 1984) and posit $\nu = U_* \ell_m |\ell_m U'(z)/U_*|^n$, where n is an integer. This reduces to $\nu = \kappa U_*[z-(n+1)h_0]$ for the logarithmic profile, and their end result for the energy-transfer coefficient is $\beta_n = \beta_0(1+\tfrac{1}{4}n)$, where β_0 is given by (1.6).

The approximation $\ell_m = \kappa(z-h_0)$ implies that the wave-induced perturbation, ν_1, of the eddy viscosity does not decay as $z\uparrow\infty$ and leads to an inhomogeneous partial differential equation for the perturbation stream function; moreover, it implies $\nu_1 = 0$ for perturbed boundary-layer flows over a level boundary. Knight's model (correctly in my view) implies homogeneous perturbation equations, although he does not display them, and that ν_1 decays exponentially. Townsend (1972) assumes the equivalent of $\ell_m = \kappa[z-h_0 e^{-kz}]$, which implies an exponential decay of ν_1 but inhomogeneous perturbation equations. I hypothesize that $\nu = \nu(z-h)$ is conserved along streamlines, which implies $\nu_1 = -\kappa U_* h$ for the logarithmic profile, where h is the vertical displacement of a streamline from its position in the basic flow. This implies that ν_1 decays exponentially and leads (in §3) to the homogeneous vorticity-transport equation (cf. (1.5))

$$D\omega/Dt = \nabla^2(\nu\omega) + 2[\nu'(U-c)]'h_{xx}, \tag{1.8}$$

wherein ω is the wave-induced vorticity, $\nu = \nu(z)$ and $U = U(z)$ are the eddy viscosity and mean velocity in the basic flow, and $\nu U' = \text{constant}$ is implicit; $\nu' = \kappa U_*$ is constant if $U(z)$ is logarithmic.

The Laplacian $\nabla^2(\nu\omega)$ in (1.8) represents diffusion and may be neglected in the limit $\epsilon \downarrow 0$ at the expense of singular behaviour at the critical layer and the relaxation of the no-slip condition at the interface. The explicit representation of ω in terms of h then reduces (1.8) to

$$\nabla \cdot [(U-c)^2 \nabla h] + 2[\nu'(U-c)]' h_x = 0, \tag{1.9}$$

which is the central result of the present analysis.

It is worth emphasizing that the last term in (1.9) rests on the hypothesis $\nu = \nu(z-h)$. This hypothesis is supported by an extension (Appendix A) of Nee & Kovasznay's (1969) eddy-viscosity-transport model or by Saffman's (1970) two-equation model (Appendix B) if diffusion is neglected in those models. This suggests that the retention of $\nabla^2(\nu\omega)$ in (1.8) may not be consistent with the hypothesis $\nu = \nu(z-h)$, but it can be shown that (1.8) does provide a consistent interpolation of the solution of (1.9) across $z = z_c$ in the limit $\epsilon \to 0$.

The elimination of the x-dependence of h through separation of variables in (1.9) yields a second-order, ordinary differential equation that is isomorphic to the Taylor–Goldstein equation for a stratified shear flow with a complex buoyancy frequency. This differential equation admits a variational integral (§5), which I use to obtain approximations to the energy-transfer coefficient β. In particular, I invoke the logarithmic profile (1.2) and the trial function $h = h_0(x)\,\mathrm{e}^{-kz}$ (which corresponds to irrotational flow) to obtain

$$\beta = \pi k z_c \left(\frac{V-c}{U_1}\right)^4 + 2\kappa^2 \left(\frac{V-c}{U_1}\right) \equiv \beta_c + \beta_\nu, \tag{1.10}$$

where V is given by (1.7) and β_c and β_ν are the critical-layer and wave-induced-Reynolds-stress components. I confirm this result through the construction of matched asymptotic expansions in Appendix C.

The total energy-transfer coefficient (1.10) agrees with the observational data compiled by Plant (1982) within the scatter of that data for $1 < c/U_* < 20$ (see figure 1 below), but the validity of the logarithmic profile in the calculation of β_c for $c/U_* \lesssim 5$ is questionable. The basic model is unreliable (for water waves) if $c/U_* \lesssim 1$, in which domain ϵ (1.3d) may not remain small; however, this domain is of limited oceanographic importance.

Kelvin–Helmholtz instability of air blowing over oil (Miles 1959b), which occurs for rather small c/U_* and for which the critical layer lies within a laminar sublayer of negligible profile curvature, should provide a good experimental test of the present model. I carry out the appropriate calculation in §7 and conclude that the available measurements (Francis 1954) are qualitatively consistent with the theory but inadequate for a quantitative test.

The assumptions of a straight-crested wave (1.1) and the mean flow $U(z)$ permit the formal averaging over turbulent fluctuations in the present model to be carried out in the transverse (y) direction without any restriction on the period of the surface wave *vis-à-vis* the spectrum of the fluctuations. But real waves and winds are not two-dimensional, and the duration of gusts may be of the order of minutes, in contrast to wave periods of the order of seconds. Whether this gustiness dominates the wind-to-wave energy transfer, as Nikolayeva & Tsimring (1986) and Janssen (1986) suggest, or whether it merely implies a slow modulation of the averages considered here, remains to be determined. The essential question is whether a particular model is useful in some significant part of the gravity-wave spectrum, and this question can be answered only through comparison with observation.

It seems likely that the principal value of the present model for practical oceanography lies in its basis (after empirical modifications) for wave-forecasting models (cf. SWAMP Group 1985). But it is worth noting that (3.7), in which the solenoidal force $-\nabla\times(\nu\omega)$ is accompanied by the Clebsch-like force $2\nu'(U-c)\,\nabla h_x$, suggests the possibility of a corresponding, non-diffusive term in other geophysical models in which diffusion is parameterized through eddy-viscosity hypotheses.

2. Kinematics

We pose the velocity field in the form (Miles 1967)

$$\{u_i\}\equiv\{u,v,w\}=[U(z-h)-c]\{1-h_z,0,h_x\}+\{u_i'(x,y,z,t)\} \tag{2.1}$$

in the Cartesian coordinates

$$\{x_i\}\equiv\{x,y,z\}\quad(i=1,2,3). \tag{2.2}$$

$U(z)$ is the mean velocity of the basic flow in a stationary reference frame, h is the y-average of the wave-induced streamline displacement, $\{u_i'\}$ is a randomly fluctuating velocity, and the subscripts x and z signify partial differentiation. By hypothesis, h is periodic in x, and

$$\bar{h}=\overline{u_i'}=\langle u_i'\rangle=0, \tag{2.3}$$

where the overbar implies an x-average and $\langle\;\rangle$ a y-average (which we subsequently designate as the *mean*). The divergence of the mean of (2.1) vanishes by construction.

The wave-induced perturbation in the mean vorticity of a particle that experiences the mean vertical displacement h from its mean elevation in the undisturbed flow is given by

$$\omega=\langle u_z-w_x\rangle-U'(z-h)=-(U-c)\nabla^2h-2U'h_z \tag{2.4\textit{a, b}}$$

to first-order in ka. In a perfect fluid this vorticity is conserved and satisfies

$$\mathrm{D}\omega/\mathrm{D}t=(U-c)\,\omega_x=-\nabla\cdot[(U-c)^2\,\nabla h_x]=0\quad(\nu=0), \tag{2.5}$$

which is equivalent to the Rayleigh equation for the stream function $\psi\equiv(U-c)h$.

An alternative formulation, which is equivalent to the present formulation to first-order in ka and maps the streamlines on lines of constant η (in particular $z=h_0$ on $\eta=0$), follows from the transformation

$$x=\xi,\quad z=\eta+h(\xi,\eta), \tag{2.6\textit{a, b}}$$

which yields

$$\langle u,w\rangle=[U(\eta)-c](1+h_\eta)^{-1}(1,h_\xi) \tag{2.7}$$

and

$$\omega=\{(1+h_\eta)^{-1}(\partial_\eta+h_\xi\partial_\eta h_\xi)-\partial_\xi h_\xi\}\left[\frac{U(\eta)-c}{1+h_\eta}\right]-U'(\eta). \tag{2.8}$$

The eddy viscosity in the perturbed flow, (3.6) below, then is $\nu=\nu(\eta)$.

3. Dynamics

Substituting the velocity field (2.1) into the momentum equations for an inviscid, incompressible fluid, averaging over y, and neglecting $O(ka)^2$, we obtain the equations of mean motion in the form (Miles 1967)

$$\mathrm{D}\langle u\rangle/\mathrm{D}t=-(U-c)^2h_{xz}=-\langle p/\rho\rangle_x-\langle u'^2\rangle_x-\langle u'w'\rangle_z\equiv X \tag{3.1\textit{a}}$$

and $$D\langle w\rangle/Dt = (U-c)^2 h_{xx} = -\langle p/\rho\rangle_z - \langle u'w'\rangle_x - \langle w'^2\rangle_z \equiv Z. \tag{3.1b}$$

The unperturbed shear flow satisfies the boundary-layer equations

$$\langle u'w'\rangle_z = 0, \quad \langle p+\rho w'^2\rangle_z = 0 \quad (ka = 0), \tag{3.2a, b}$$

by virtue of which (3.1 a, b) are first order in ka.

We now invoke the closure hypothesis (cf. Boussinesq 1877)

$$\langle u_i' u_j'\rangle = \tfrac{1}{3}\delta_{ij}\langle u_k' u_k'\rangle - \nu\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right), \tag{3.3}$$

wherein repeated indices are summed over 1–3 and ν is the eddy viscosity†, and introduce

$$\pi = \langle p/\rho + \tfrac{1}{3}u_i' u_i'\rangle. \tag{3.4}$$

(The rate at which work is done on the wave by the normal aerodynamic stress is $-(U-c)\overline{\pi h_x}$ per unit area, by virtue of which π is the relevant kinematic pressure in the present context.) The eddy viscosity in the unperturbed flow, $\nu(z)$, is determined by the requirement that the shear stress be constant through the boundary layer:

$$(\nu U')' = 0 \quad (ka = 0). \tag{3.5}$$

Phenomenological arguments (cf. Goldstein 1938, §§80–85) suggest, and I assume, that ν is conserved along streamlines, and therefore may be approximated by (see also Appendix A)

$$\nu = \nu(z-h) = \nu(z) - \nu'(z)\,h, \tag{3.6}$$

in the limit $k\nu/V \downarrow 0$. Combining (2.1) and (3.3)–(3.6) on the right-hand side of (3.1) and introducing the vorticity ω from (2.4b), we resolve (X, Z) into solenoidal and Clebsch-like (cf. Lamb 1932, §167) components according to

$$(X, Z) = -\boldsymbol{\nabla}\pi - \boldsymbol{\nabla}\times(\nu\boldsymbol{\omega}) + 2\nu'(U-c)\,\boldsymbol{\nabla}h_x, \tag{3.7}$$

where $\boldsymbol{\nabla} \equiv (\partial_x, \partial_z), \boldsymbol{\omega} \equiv \boldsymbol{y}_1\,\omega$, $\boldsymbol{y}_1$ is a unit vector in the y-direction, and, here and subsequently, $\nu = \nu(z)$ is the eddy viscosity in the unperturbed flow. Substituting (3.7) into (3.1) and eliminating π, we obtain the vorticity-transport equation (cf. (2.5))

$$D\omega/Dt = \nabla^2(\nu\omega) + 2[\nu'(U-c)]' h_{xx}, \tag{3.8}$$

in which $\nabla^2(\nu\omega)$, which is derived from the solenoidal component of (3.7), represents diffusion (its form, *vis-à-vis* $\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\omega)$, follows from the definition (2.4) and the constraint (3.5)), and the last term, which is derived from $2\nu'(U-c)\,\boldsymbol{\nabla}h_x$, represents vorticity transfer between the basic and wave-induced flows. We remark that, owing to the invocation of $\nu U'' \equiv -\nu' U'$ in its derivation, (3.8) is equivalent to the Orr–Sommerfeld equation only if both U' and ν are constant.

It follows from the assumptions of monochromatic motion (1.1) and linearity that h, ω and π admit the representation

$$[h, \omega, \pi] = a\,\mathrm{Re}\{e^{ikx}[H(\zeta), \quad k^2 V\Omega(\zeta), \quad kV^2\Pi(\zeta)]\}, \qquad \zeta \equiv kz, \tag{3.9a, b}$$

where V is a reference velocity, and H, Ω and Π are dimensionless, complex amplitudes. Combining (2.4b), (3.1) and (3.7)–(3.9) and introducing‡

$$w(\zeta) \equiv \frac{U-c}{V}, \quad \lambda \equiv \frac{k\nu}{V}, \tag{3.10a, b}$$

we obtain $$\Pi = w^2 H' + 2i\lambda' wH - i(\lambda\Omega)', \quad \Pi' = w^2 H + 2i\lambda' wH' - i\lambda\Omega, \tag{3.11a, b}$$

$$\Delta(\lambda\Omega) - iw\Omega - 2(\lambda' w)' H = 0, \quad \Omega = -w\,\Delta H - 2w'H', \tag{3.12a, b}$$

† The molecular viscosity, which may be significant in the laminar sublayer (wherein ν and U' are constant), may be included in $\nu(z)$.

‡ The definition (3.10a) for w holds throughout the subsequent development, in which the dimensional vertical velocity of §2 no longer appears.

where, here and subsequently, $' \equiv \mathrm{d}/\mathrm{d}\zeta$ and $\Delta \equiv (\mathrm{d}/\mathrm{d}\zeta)^2 - 1$. The corresponding boundary conditions, which follow from continuity of the interfacial velocity (we neglect surface drift, so that the horizontal velocity induced by the wave is kch_0) and the requirement of finite energy, are

$$H = 1, \quad H' = 1 \quad (\zeta \downarrow 0), \quad H \to 0 \quad (\zeta \uparrow \infty). \tag{3.13a–c}$$

4. The reduced equations

The solenoidal component of (3.7), and hence also $\nabla^2(\nu\omega)$ in (3.8), may be neglected in the limit $\epsilon \downarrow 0$ at the expense of singular behaviour at the critical layer, the relaxation of the no-slip condition (3.13*b*), and an $O(\epsilon^2)$ error in Π. The corresponding reduction of (3.8) yields (1.9), while that of (3.11)–(3.13) yields

$$\Pi = w^2H' + 2\mathrm{i}\lambda' wH, \quad \Pi' = w^2H + 2\mathrm{i}\lambda' wH', \tag{4.1a, b}$$

$$(w^2H')' - [w^2 - 2\mathrm{i}(\lambda' w)']\,H = 0, \tag{4.2}$$

and

$$H = 1 \quad (\zeta = 0), \qquad H \to 0 \quad (\zeta \uparrow \infty). \tag{4.3a, b}$$

We remark that (4.2) is formally equivalent to a modified (Miles 1961) Taylor–Goldstein equation (Drazin & Reid 1981, §44.1) for a stratified shear flow with a complex buoyancy frequency. But note that, in contrast to the buoyancy force in a stratified flow, the pseudo-buoyancy force in (4.1) comprises both horizontal and vertical components and is in phase with the particle velocity rather than the particle displacement.

If $\lambda' = 0$ (4.2) is equivalent to Rayleigh's equation, to which it reduces through the transformation $\phi = wH$. The exponents of the regular singularity at $w = 0$ are $-1 + 2\mathrm{i}\delta$ and $-2\mathrm{i}\delta$, where $\delta \equiv 2\lambda'/w_c' = 2\kappa^2\zeta_c \ll 1$, and may be approximated by -1 and 0 (0 and 1 for Rayleigh's equation) in the present context. This approximation fails in $|\zeta - \zeta_c| = O(\kappa^2\zeta_c^2)$, but the resulting error is within that already implicit in (4.2).

Diffusion may be incorporated, as in the asymptotic solution of the Orr–Sommerfeld equation, by separating the solution of (3.12) and (3.13) into H_1, a solution of (4.2), for which the lengthscale is 1 ($1/k$ for z), and H_2, a solution of the truncated vorticity equations

$$(\lambda\Omega)'' = \mathrm{i}w\Omega, \quad \Omega = -wH'' - 2w'H', \tag{4.4a, b}$$

for which the lengthscale is $\epsilon^{\frac{3}{2}}/k$. The contribution of H_2 to Π is $O[\epsilon^3(U_0'/kc)^{\frac{1}{2}}]$ and therefore negligible in the present approximation. H_2 does introduce an interfacial shear stress, but the resulting dissipation is negligible compared with that in the water (cf. Miles 1959*a*).

5. Variational formulation

We require $\Pi_0 \equiv \Pi_r + \mathrm{i}\Pi_i$, as determined from (4.1*a*) through the solution of (4.2) and (4.3). The imaginary part Π_i, to which the energy transfer to the wave is proportional, comprises contributions from the critical layer, $\zeta = \zeta_c$, and from the wave-induced Reynolds stresses. These contributions are small and may be calculated independently by neglecting $(\lambda' w)'$ in (4.2) in the calculation of the critical-layer component Π_c and neglecting the singularity at $\zeta = \zeta_c$ (with all integrals regarded as principal values) in the calculation of $\Pi_0 - \Pi_c$. Moreover, since $\zeta_c \ll 1$, (1.3*b*) in the present formulation, Π_c may be calculated from Π_r according to (Miles 1959*a*, Appendix B, wherein $\alpha + \mathrm{i}\beta \equiv \Pi/\epsilon^2$)

$$\Pi_c = \mathrm{i}\pi\epsilon^{-2}\zeta_c\,\Pi_r^2[1 + O(\zeta_c^2)]. \tag{5.1}$$

It then remains to calculate $\Pi_0 - \Pi_c$, for which we proceed to obtain a variational approximation.† The direct calculation of Π_0 through the matched-asymptotic solution of (4.2) and (4.3) is carried out in Appendix C.

Multiplying (4.2) through by H, integrating over $(0, \infty)$ by parts and invoking (4.1 *a*) and (4.3 *a*, *b*), we obtain the variational integral (cf. Miles 1959*b*)

$$\Pi_0 \equiv \Pi_r + \mathrm{i}\Pi_i = -\int_0^\infty [w^2(H'^2 + H^2) + 4\mathrm{i}\lambda' wHH']\, d\zeta, \tag{5.2}$$

which is stationary with respect to first-order variations of H about the true solution of (4.2) and (4.3).

A simple trial function for the determination of $\Pi_0 - \Pi_c$ is provided by the solution of (4.2) and (4.3) for constant λ and w (irrotational flow),

$$H = \mathrm{e}^{-\zeta}[1 + O(\epsilon)]. \tag{5.3}$$

Substituting (5.3) into (5.2) and invoking the variational principle that the error in the variational integral is of the order of the square of the error in the trial function, we obtain

$$\Pi_0 - \Pi_c = \int_0^\infty (-2w^2 + 4\mathrm{i}\lambda' w)\,\mathrm{e}^{-2\zeta}\, d\zeta + O(\epsilon^2). \tag{5.4}$$

Evaluating the integral for the logarithmic profile

$$w = \epsilon \log(\zeta/\zeta_c), \quad \lambda' = \epsilon\kappa^2, \quad \epsilon = U_1/V \ll 1, \tag{5.5a–c}$$

which follow from (1.2) and (3.10), we reduce (5.4) to

$$\Pi_0 - \Pi_c = -w_1^2 + 2\mathrm{i}\lambda' w_1 + O(\epsilon^2), \tag{5.6a}$$

where

$$w_1 \equiv \epsilon \log(\zeta_1/\zeta_c) = O(1), \quad \zeta_1 = \tfrac{1}{2}\mathrm{e}^{-\gamma} = 0.281. \tag{5.6b, c}$$

The trial function (5.3) does not comprehend the singularity at $\zeta = \zeta_c$ and therefore fails to give Π_c; however, it follows from (5.1) and (5.6*a*) that

$$\Pi_c = \mathrm{i}\pi\epsilon^{-2}\zeta_c w_1^4 = \mathrm{i}\pi\epsilon^2\zeta_c \ln^4(\zeta_1/\zeta_c), \tag{5.7}$$

and hence (after invoking $\lambda' = \epsilon\kappa^2$) that the total imaginary part of Π is given by

$$\Pi_i = \epsilon^2[\pi\zeta_c \ln^4(\zeta_1/\zeta_c) + 2\kappa^2 \ln(\zeta_1/\zeta_c)]. \tag{5.8}$$

We remark that the $O(\epsilon^2)$ error in (5.4) is formally equivalent to that implicit in the reduced equations of §4 and the presumed independence of critical-layer and wave-induced-Reynolds-stress components of Π_i; accordingly, the improvement of the variational approximation (5.4) through a more flexible trial function (cf. Miles 1959*b*) does not appear to be profitable.

6. Comparison with observation

If the reference velocity is taken to be U_1 (as in I), rather than V, the energy-transfer parameter Π_i is replaced by (cf. I)

$$\beta = \epsilon^{-2}\Pi_i \equiv \beta_c + \beta_\nu, \tag{6.1}$$

where

$$\beta_c \equiv \pi\zeta_c \ln^4(\zeta_1/\zeta_c), \quad \beta_\nu = 2\kappa^2 \ln(\zeta_1/\zeta_c) \quad (\zeta_c \ll 1). \tag{6.2a, b}$$

We restrict further consideration to fully developed rough flow, for which (I, §7)

$$gz_0/U_1^2 = \Omega, \quad \zeta_c = \Omega(U_1/c)^2 \mathrm{e}^{c/U_1}, \tag{6.3a, b}$$

† Π_c may be calculated from a variational integral for $\phi = wH$ (which tends to a constant at $\zeta = \zeta_c$), as in I, §4, but the singularity in H is too strong to permit such a calculation from (5.2).

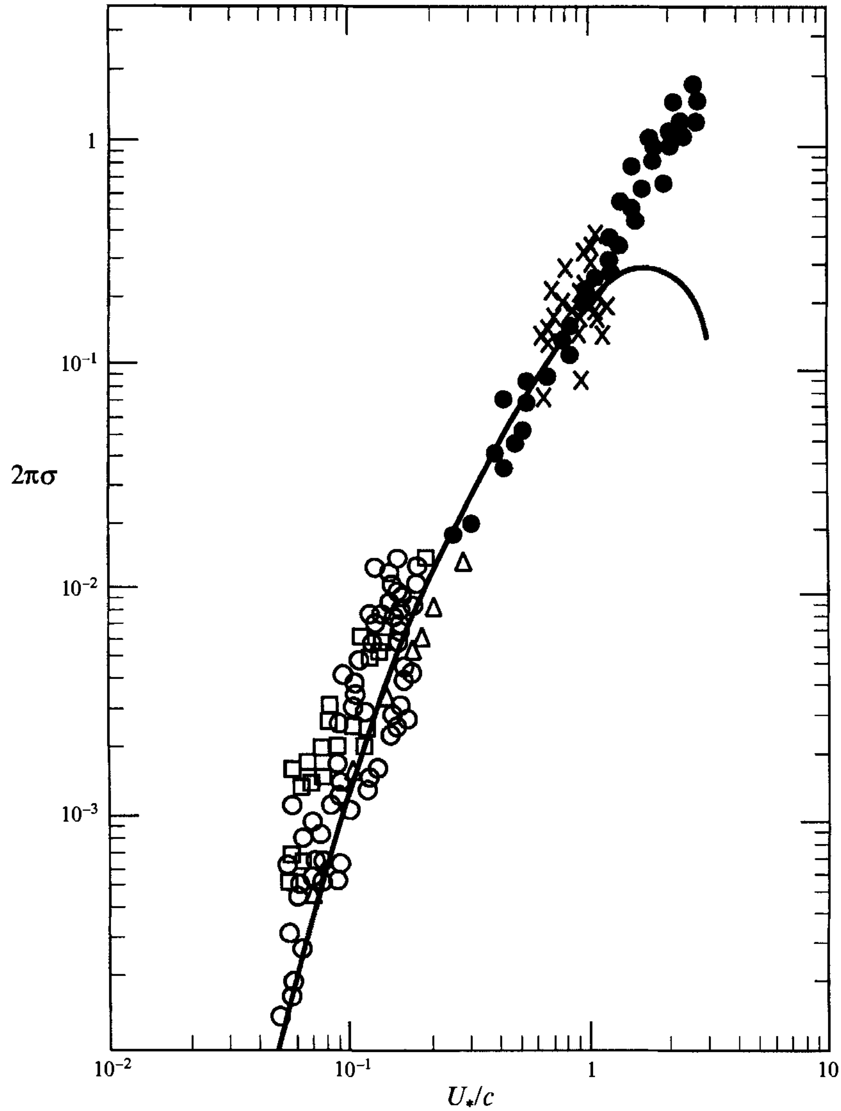

FIGURE 1. The dimensionless growth rate $2\pi\sigma = 2\pi s\beta(U_1/c)^2$, as calculated from (6.1)–(6.3) for $\kappa = 0.4$, $\Omega = 2.3 \times 10^{-3}$ and $s = 1.25 \times 10^{-3}$, compared with Plant's (1982) compilation of observational data. The decay of the theoretical result in $U_*/c \gtrsim 1$ is associated with (6.3).

where $\Omega = O(10^{-3}\text{–}10^{-2})$ is Charnock's (1955) constant. Each of β_c, β_v and $\beta_c + \beta_v$ has a maximum at $c/U_1 = 2$ if $\Omega > 2.8 \times 10^{-3}$. The maximum of β_c for $\Omega = 10^{-2}$ is 3.39, which compares with the maximum of 3.32 obtained through numerical integration of the Rayleigh equation (Miles 1959*a*). The corresponding maximum of β_v (for $\kappa = 0.4$) is 0.87. We emphasize that the validity of the logarithmic profile for c/U_1 as small as 2 is questionable, even for rough flow (cf. Riley & Donelan 1982), and (6.2) may overestimate β_c for $c/U_1 \lesssim 2$.

The growth-rate parameter $2\pi\sigma = 2\pi s\beta(U_1/c)^2$ (1.4*a*) is compared with Plant's (1982) compilation of laboratory and field data for $0.05 \leqslant U_*/c \leqslant 3$ in figure 1, following van Duin & Janssen (1992), who use ξ and ζ_r for the present β and σ and choose $\kappa = 0.4$, $s = 1.25 \times 10^{-3}$ and $\Omega = 2.3 \times 10^{-3}$. The agreement is within the scatter of the data for $c/U_* \lesssim 1$, but the theoretical result is compromised by the possible overestimate of profile curvature for $c/U_* \lesssim 5$. The corresponding results for $\Omega = 10^{-2}$ are inferior by at most a factor of 2 to, and for $\Omega = 10^{-3}$ are indistinguishable from, those of figure 1 in $U_*/c \lesssim 1$. Indeed, the difference between the result in figure 1 and that based on β_ν alone, rather than $\beta_\nu + \beta_c$, is within the scatter of the data for $0.07 \leqslant U_*/c \leqslant 1$ (in which range β_ν varies from 0.42 to 0.74 and β_c from 0.70 to 2.50). But the result based on β_ν alone is significantly below the data; cf. the curve for $n = 0$ in van Duin & Janssen's (1992) figure 1, which needs to be lowered by roughly a factor of 2 to account for the replacement of $V = U(1/k)$ by $U(z_1)$.

7. Kelvin–Helmholtz instability

Kelvin–Helmholtz instability of the interface between a parallel shear flow of a light, inviscid fluid and a viscous liquid (e.g. air blowing over oil) occurs for $c = 0$ if the wave-induced Reynolds stresses are neglected (Miles 1959*b*) and may be expected to occur for rather small c if they are of the magnitude implied by the present model. The dynamical equilibrium between the aerodynamic pressure (3.9) and the pressure induced at the interface by the wave (1.1) moving with the wave speed c (Lamb 1932, §349) is

$$\rho_-\left\{c^2 - c_0^2 + (2\kappa\nu_-)^2\left[\left(1-\frac{\mathrm{i}c}{k\nu_-}\right)^{\frac{1}{2}} - 1 + \frac{\mathrm{i}c}{k\nu_-}\right]\right\} = \rho_+ V^2 \Pi_0, \tag{7.1}$$

where

$$c_0^2 = gk^{-1} + Tk, \tag{7.2}$$

$\rho_\pm$ is the density of the upper/lower fluid, ν_- is the kinematic viscosity of the lower fluid, and T is the kinematic surface tension (neglected in the preceding formulation).

We proceed on the hypothesis that

$$0 < c \ll k\nu_-, U_1, \tag{7.3}$$

adopt the approximation (5.6) for Π_0, and neglect Π_c by virtue of $c \ll U_1$. Expanding the left-hand side of (7.1) in powers of $c/k\nu_-$ and restoring dimensions in (5.6), we obtain

$$-c_0^2 + 2\mathrm{i}k\nu_- c + O(c^2) = s(-V^2 + 2\mathrm{i}\kappa U_* V)[1 + O(\epsilon^2)], \quad s \equiv \rho_+/\rho_-. \tag{7.4a, b}$$

The real and imaginary parts of (7.4) imply

$$c_0^2 = sV^2, \quad c = s(\kappa U_*/k\nu_-)V. \tag{7.5a, b}$$

The critical wind speed and wavenumber are determined by the requirement that $V = V(k)$ be a minimum; however, in testing the present aerodynamic model, it may be preferable to invoke the measured values of k and U_* and compare the measured value of c with

$$c = s^{\frac{1}{2}}(\kappa U_*/k\nu_-)\,c_0. \tag{7.6}$$

Francis (1954) observed air blowing over an oil for which $\rho_- = 0.875$ gm/cm^3, $\rho_- T = 34$ dynes/cm and $\nu_- = 2.5$ cm^2/s and measured the critical values $U_1 = 97$ cm/s and $\lambda = 2\pi/k = 2$ cm. These compare with the predictions $U_1 = 93$ cm/s and $\lambda = 1.8$ cm from a variational improvement of (7.5*a*) (Miles 1959*b*). Francis's 'estimated' wave speed was 'about 1 cm/s', which compares with $c = 1.4$ cm/s from (7.6), but he subsequently measured a surface drift of 0.4 cm/s (private communication), which would imply an observed c of 0.6 cm/s. Given the uncertainty in the observed value of c and the possibility of an increase in the effective value of ν_-

through surface contamination (although this presumably is less important for oil than for water), the test of the present theory provided by Francis's experiments is at best marginal, and it would be desirable to have more precise measurements of c for somewhat smaller ν_-.

I am indebted to C. A. van Duin, P. A. E. M. Janssen and W. J. Plant for the data in figure 1, to Paul Libby for tutorials on turbulent modelling, and to Nicholas Rott for (many years ago) showing me the virtues of focusing on vorticity. This work was supported in part by the Division of Ocean Sciences of the National Science Foundation, NSF Grant OCE-92-16397, and by the Office of Naval Research N00014-92-J-1171.

Appendix A. Eddy-viscosity-transport model

Following Nee & Kovasznay (1969), but neglecting molecular viscosity and diffusion in keeping with the reduced equations of §4, we posit the phenomenological transport equation

$$\mathrm{D}\nu/\mathrm{D}t \equiv (\partial t + u_i \partial x_i)\,\nu = AS\nu - (B/\ell^2)\,\nu^2, \tag{A 1}$$

wherein the first and second terms on the right-hand side represent, respectively, generation and decay, A and B are constants, S is a scalar invariant of the velocity-gradient tensor

$$u_{ij} \equiv \partial\langle u_i\rangle/\partial x_j, \tag{A 2}$$

and ℓ is a local lengthscale. Nee & Kovasznay choose (with minor differences in notation and normalization) $S = |U'(z)|$ and $\ell = \kappa z$, infer $B = A$ from the requirement that (A 1) be satisfied by $U' = U_*/\kappa z$ and $\nu = \kappa U_* z$ (the logarithmic boundary layer), and estimate $A = 0.13$.

Appropriate generalizations of $S = |U'|$ and $\ell = \kappa z$ in the present context are the magnitude of the mean vorticity (cf. Spalart & Allmaras 1992) and the mixing length constructed in §1:

$$S = |\langle u_z - w_x\rangle| = |U'(z-h) + \omega|, \quad \ell = \kappa(z-h). \tag{A 3a, b}$$

Substituting (A 3) and $B = A$ into (A 1), neglecting $\nu\omega$, which is consistent with the neglect of diffusion in §4, and invoking $U' > 0$, we obtain

$$\frac{\mathrm{D}\nu}{\mathrm{D}t} = A\left\{U'(z-h)\,\nu - \left[\frac{\nu}{\kappa(z-h)}\right]^2\right\}. \tag{A 4}$$

This is satisfied without further approximation by $\nu = \nu(z-h) = \kappa^2(z-h)^2 U'(z-h)$, which is compatible with the assumption of constant stress in the basic flow, $\nu U' = U_*^2 = \text{constant}$, only for the logarithmic boundary layer.

Plausible alternatives to (A 3a) for S are the scalar norms of the velocity-gradient tensor, u_{ij}, and the rate-of-strain tensor, $\epsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$:

$$S = (u_{ij}u_{ij})^{\frac{1}{2}} \approx U' - [(U-c)\,h]_{zz} + O(h^2) \tag{A 5a}$$

$$= U' + \omega - U''h + (U-c)\,h_{xx}, \tag{A 5b}$$

and

$$S = (2\epsilon_{ij}\epsilon_{ij})^{\frac{1}{2}} \approx U' - [(U-c)\,h]_{zz} + (U-c)\,h_{xx} + O(h^2) \tag{A 6a}$$

$$= U' + \omega - U''h + 2(U-c)\,h_{xx}, \tag{A 6b}$$

wherein u_{ij}, as defined by (A 2), has been calculated from (2.1) and (A 5/6b) follows from (A 5/6a) through (2.4). Combining either (A 5b) or (A 6b) with (A 3b) and the linear perturbations

$$\nu = \nu_0 + \nu_1, \quad \mathrm{D}\nu/\mathrm{D}t \approx (U-c)\,\nu_{1x} + w\nu_0' = (U-c)(\nu_1 + \nu_0' h)_x, \qquad \text{(A 7}a, b\text{)}$$

and

$$\frac{\nu^2}{\ell^2} \approx \left(\frac{\nu_0}{\kappa z}\right)^2 \left(1 + \frac{2\nu_1}{\nu_0} + \frac{2h}{z}\right) = U_*^2 + 2U'(\nu_1 + \nu_0' h) \qquad \text{(A 8)}$$

in (A 1), neglecting $\nu_0 \omega$ (as above), and invoking $\nu_0 U'' = -\nu_0' U'$ ($\nu_0 U' = U_*^2 =$ constant), we obtain

$$[(U-c)\,\partial_x + AU'](\nu_1 + \nu_0' h) = nA\nu_0 (U-c)\, h_{xx}, \qquad \text{(A 9)}$$

where $n = 1$ for S (A 5) or 2 for S (A 6).

The integration of (A 9) yields

$$\nu_1 = -\nu_0' h + nA\nu_0 \left[h_x - \kappa \int_{-\infty\,\mathrm{sgn}\,\kappa}^{x} \mathrm{e}^{\kappa(\xi - x)} h_\xi(\xi, z)\,\mathrm{d}\xi \right], \quad \kappa \equiv \frac{AU'}{U-c}. \qquad \text{(A 10}a, b\text{)}$$

The dominant component, $-\nu_0' h$, corresponds to the approximation (3.6). The component $nA\nu_0 h_x$ does not contribute to the energy transfer to the surface wave and makes only an $O(\epsilon^2 A)$ contribution to the aerodynamic inertia. The remaining integral is negligible in the present approximation.

Appendix B. Wave-induced viscosity from Saffman's model

It may be objected that the assumption of $\ell = \kappa(z-h)$, for the lengthscale in (A 1) is tantamount to the approximation (3.6). This criticism, which is basically applicable to any one-equation transport model in which a local lengthscale is hypothesized, may be addressed by considering a two-equation model. Following Knight (1977), we consider Saffman's (1970) model, in which

$$\nu = e/\eta, \qquad \text{(B 1)}$$

e and η (Saffman's ω) satisfy the transport equations

$$\frac{\mathrm{D}e}{\mathrm{D}t} = \Gamma e - e\eta + \bar{\sigma}\nabla\cdot(\nu\nabla e), \quad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \Delta\eta - \beta\eta^2 + \sigma\nabla\cdot(\nu\nabla\eta), \qquad \text{(B 2}a, b\text{)}$$

$$\Gamma = \bar{\alpha}(2\epsilon_{ij}\epsilon_{ij})^{\frac{1}{2}}, \quad \epsilon_{ij} = \tfrac{1}{2}(u_{ij} + u_{ji}), \quad \Delta = \alpha(u_{ij}u_{ij})^{\frac{1}{2}}, \qquad \text{(B 3}a\text{–}c\text{)}$$

u_{ij} is defined by (A 2), and α, $\bar{\alpha}$, β, σ and $\bar{\sigma}$ are empirical constants. Substituting $e = \nu\eta$ from (B 1) into (B 2a), subtracting $\nu \times$(B 2b) from the result, and neglecting the diffusion terms (as is consistent with the reduced equations in §4), we obtain

$$\frac{\mathrm{D}\nu}{\mathrm{D}t} = (\Gamma - \Delta)\,\nu + (\beta - 1)\,\eta\nu = 0, \quad \frac{\mathrm{D}\eta}{\mathrm{D}t} = \Delta\eta - \beta\eta^2. \qquad \text{(B 4}a, b\text{)}$$

It can be shown that $(\beta-1)\,\eta\nu$ is negligible in (B 4a) in the present approximation for any $\beta - 1 = O(1)$; however, for simplicity, we choose $\beta = 1$ (Knight gives $\frac{5}{6} < \beta < 1$).

We perturb the basic flow, for which

$$[u_i] = [U(z), 0, 0], \quad e_0 = \bar{\alpha}U_*^2, \quad \eta_0 = \bar{\alpha}U', \quad \nu_0 = U_*^2/U', \qquad \text{(B 5}a\text{–}d\text{)}$$

as in (A 5)–(A 7) to obtain (cf. (A 9))

$$[(U-c)\,\partial_x - (\bar{\alpha} - \alpha)\,U'](\nu_1 + \nu_0' h) = (2\bar{\alpha} - \alpha)\,\nu_o (U-c)\,h_{xx}, \qquad \text{(B 6)}$$

the integration of which yields

$$\nu_1 = -\nu_0' h + (2\bar{\alpha} - \alpha)\nu_0 h_x + \dots . \tag{B 7}$$

The coefficient $2\bar{\alpha}-\alpha = 0.5$ (using Knight's $\bar{\alpha} = 0.3$ and $\alpha = 0.1$) compares with $nA = 0.13/0.26$ for $n = 1/2$ in (A 10). The component $-\nu_0' h$ in (B 7) agrees with that inferred from the assumption (3.5) that ν is conserved along streamlines. The component $(2\bar{\alpha}-\alpha)\nu_0 h_x$, although of the same order of magnitude as $\nu_0' h$ for $kz = O(1)$, does not contribute to the energy transfer to the surface wave and makes only an $O(\epsilon^2)$ contribution to the aerodynamic inertia.

The result (B 6), with slightly different constants, also may be derived from Jones & Launder's (1972) two-equation model.

Appendix C. Matched-asymptotic solution of reduced equations

We proceed on the assumption that w (5.5*a*) admits the representation

$$w = w_1(1+\epsilon_1 \ell), \quad w_1 \equiv \epsilon \log(\zeta_1/\zeta_c) = O(1), \quad \ell \equiv \log(\zeta/\zeta_1) = O(1), \quad \epsilon_1 \equiv \epsilon/w_1, \tag{C 1a–d}$$

where $\zeta_1 = O(1)$ is a scaling parameter for which, anticipating the simplification of the subsequent analysis, we choose the value $\frac{1}{2}e^{-\gamma}$ (5.6*c*). (The end results are independent of the initial choice of ζ_1.)

We pose the outer approximation to the solution of (4.2) and (4.3*b*) in the form

$$w(\zeta)\, H(\zeta) = \Phi(\zeta)\, e^{-\zeta} \tag{C 2}$$

and transform (4.2) to the integral equation (cf. Lighthill 1957)

$$\Phi(\zeta) = \Phi_\infty + \frac{1}{2}\int_\zeta^\infty [1-e^{-2(\eta-\zeta)}]\, q(\eta)\, \Phi(\eta)\, d\eta, \quad q = w^{-1}[w'' - 2i(\lambda' w)' w^{-1}], \tag{C 3a, b}$$

in which Φ_∞ is a constant and the integral is $O(\epsilon)$. The path of integration is indented under the singular point at $\zeta = \zeta_c \ll 1$. Substituting w from (C 1) into (C 3*b*) and solving by iteration, starting from $\Phi = \Phi_\infty + O(\epsilon)$, we obtain the successive approximations

$$\Phi/\Phi_\infty = 1 - \epsilon_1[\mathscr{E}(\zeta) + i\kappa^2] + O(\epsilon^2) \tag{C 4}$$

and

$$\Phi/\Phi_\infty = 1 - \epsilon_1[\mathscr{E} + i\kappa^2] + \epsilon_1^2[\mathscr{E}^2 + \mathscr{E}\ell + \mathscr{F} + i\kappa^2\mathscr{E} - \kappa^4] + O(\epsilon^3), \tag{C 5}$$

where

$$\mathscr{E}(\zeta) \equiv \int_\zeta^\infty e^{-2(\eta-\zeta)}\eta^{-1}\, d\eta = e^{2\zeta}E_1(2\zeta), \quad \mathscr{F}(\zeta) = \int_\zeta^\infty [2 + e^{-2(\eta-\zeta)}]\, \mathscr{E}^2(\eta)\, d\eta, \tag{C 6a, b}$$

and E_1 is an exponential integral (Abramowitz & Stegun 1964, §5.2).

Turning to the inner domain, $\zeta = O(\epsilon)$, we transform (4.2) and (4.3*a*) to

$$H(\zeta) = 1 + w_0^2 H_0' K(\zeta) + \int_0^\zeta \frac{d\eta}{w^2(\eta)} \int_0^\eta \{w^2(\xi) - 2i[\lambda'(\xi)\, w(\xi)]'\}\, H(\xi)\, d\xi, \tag{C 7}$$

where

$$K \equiv \int_0^\zeta \frac{d\eta}{w^2(\eta)} = -\frac{1}{w'w} + \frac{1}{\epsilon w_c'}\,\mathrm{Ei}\left(\frac{w}{\epsilon}\right) - i\pi \frac{w_c''}{w_c'^3}\mathscr{H}(\zeta - \zeta_c), \tag{C 8a}$$

$$\sim \frac{\zeta}{w^2}\left(1 + \frac{2\epsilon}{w} + \dots\right) - i\pi\frac{w_c''}{w_c'^3} \quad (w/\epsilon \uparrow \infty), \tag{C 8b}$$

where Ei is an exponential integral (Abramowitz & Stegun 1964, §5.1), and $\mathscr{H}$ is Heaviside's step function. Solving (C 7) by iteration, starting from $H = 1 + O(\epsilon)$, we obtain the successive approximations

$$H = 1 + w_0^2 H_0' K(\zeta) + O(\epsilon^2) \tag{C 9}$$

and
$$H = 1 + \Pi_0 K(\zeta) + \int_0^\zeta \frac{\mathrm{d}\eta}{w^2(\eta)} \int_0^\eta w^2(\xi)\,\mathrm{d}\xi - 2\mathrm{i} \int_0^\zeta \frac{\lambda'(\eta)\,\mathrm{d}\eta}{w(\eta)} + O(\epsilon^3), \tag{C 10}$$

where
$$\Pi_0 = w_0^2 H_0' + 2\mathrm{i}\lambda_0' w_0. \tag{C 11}$$

Matching the inner expansion of the outer approximation to H provided by (C 2) and (C 5) to the outer expansion of the inner approximation (C 10) and invoking (C 1), we obtain

$$\Phi_\infty = w_1[1 + \mathrm{i}(\epsilon_1 \kappa^2 - \pi\epsilon^{-2}\zeta_c w_1^2) - \epsilon_1^2 \mathscr{F}_0 + O(\epsilon^3)] \tag{C 12}$$

and
$$\Pi_0 = -w_1^2 + \mathrm{i}(\pi\epsilon^{-2}\zeta_c w_1^4 + 2\epsilon\kappa^2 w_1) + O(\epsilon^2), \tag{C 13}$$

which agrees with the variational approximation in §5.

Combining (C 2), (C 4), (C 8), (C 9), (C 12) and (C 13), we obtain the composite approximation

$$H = \{1 - \epsilon_1[\mathscr{E}(\zeta) + \ell] - w_1^2 K(\zeta) + \zeta + 2\epsilon_1 \zeta(1-\ell) + O(\epsilon^2)\}\,\mathrm{e}^{-\zeta}. \tag{C 14}$$

Note added in proof

Dr van Duin (private communication) does not agree with my statement, following (1.7*a, b*) above, that he and Janssen (1992) neglect the energy transfer associated with the phase shift across $U = c$. He argues that 'on the basis of the eddy viscosity model, there is no such energy transfer [since] there is no phase shift because of the strong effect of the turbulence'. This argument appears to reflect the transcendental smallness of kz_c implied by their asymptotic scaling.

Belcher & Hunt (1993) invoke a similar scaling but use a different model for the wave-induced Reynolds stresses. They conclude that 'The asymptotic theory of van Duin & Janssen (1992) [and that of Jacobs (1987) and, implicitly, the present calculation of β_ν] leads to a growth rate that is a factor of $O(1/\epsilon)$ too large.'

These conflicts merit further investigation.

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